Stability Properties of a Heat Equation with State-Dependent Parameters and Asymmetric Boundary Conditions

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Stability properties of a heat equation with state-dependent parameters and asymmetric boundary conditions

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Abstract: In this work the stability properties of a partial differential equation (PDE) with state-dependent parameters and asymmetric boundary conditions are investigated. The PDE describes the temperature distribution inside foodstuff, but can also hold for other applications and phenomena. We show that the PDE converges to a stationary solution given by (fixed) boundary conditions which explicitly diverge from each other. Numerical simulations illustrate the results.

Keywords: Parabolic PDE, heat equation, stability analysis, state-dependent parameters

1. INTRODUCTION

In order to extend shelf life of different foodstuff, freezing has shown itself superior to many other preservation techniques. Among other reasons, freezing preserves distinct characteristics of the original product to a large extent, like e.g. taste and nutritional value. If suitable freezing and storage methods are applied correctly, food can be stored for months or even years without significant degradation of specific quality characteristics. Especially for rapidly spoiling food, such as fish and fish products, freezing is often an essential part in the supply chain to deliver high-quality and safe products to the consumer.

As the temperature-dynamics during freezing depend both on space and time, a good approach to model these dynamics is using distributed parameter systems (DPS), in particular partial differential equations (PDEs). The most common PDE for thermal problems is the so called heat equation, a parabolic PDE. The heat equation as a model for freezing problems has been extensively studied, e.g. in Pham (2006a), Pham (2006b) and Woinet et al. (1998). We present additional publications concerning the modeling of heat and mass transfer in (frozen) foodstuff in Backi et al. (2014b). In the same article we give an overview over different articles and books concerning (stability-) analyses of classes of PDE-systems. Most of them study the Burgers' equation and its potential form, since we showed a connection between these equations and a heat equation with state-dependent parameter functions. In addition we want to point to the works of Hopf (1950) and Cole (1951), who attend the topic of transformations between PDEs, in particular the Burgers' equation. Furthermore, we refer to publications concerning stability (and to some degree also control) of infinite-dimensional systems and PDEs in general. Especially we want to highlight the work of Dashkovskiy and Mironchenko (2013), Krstic and Smyshlyaev (2008) and Smyshlyaev and Krstic (2010).

A heat equation, studied in Backi et al. (2014b), is derived from the diffusion equation with temperature-dependent parameter functions. The diffusion equation can in some special cases describe transport phenomena, as for example pointed out in Hasan et al. (2010). The standard heat equation does not permit modeling for phase change phenomena, such as *thermal arrest* caused by *latent heat of fusion*. These phenomena have to be imposed to the PDE, e.g. by adapting the parameter functions. In the present case this is done by applying the so called *apparent heat capacity method* as introduced e.g. in Muhieddine et al. (2008), which in principle overestimates the specific heat capacity $c(T)$ in a region $I_{\Delta T} = T_F \pm \Delta T$ around the freezing temperature T_F in order to slow down heat transfer.

The problem considered in this paper is somewhat related to the classical Stefan problem as it can be interpreted as a "2-sided Stefan problem without insulated boundaries". However, the Stefan problem consists of "sharp switching" between two linear heat equations with constant parameters, whereas the problem considered here consists of one heat equation with state-dependent parameters resulting in an "intermediate zone" around the freezing point.

As pointed out in Backi et al. (2014b) the explicit definition of the parameters as state-dependent functions makes a stability investigation necessary as already established results for the linear heat equation and the (Potential) Burgers' equation cannot be applied directly. The present work provides an extension to the proof in Backi et al. (2014b), where assumptions on the parameter functions had to be imposed in order to prove stability for the heat equation with symmetric boundary conditions. The question of necessity of a stability analysis for this kind of PDE might arise, as it models the freezing of foodstuff. Freezing processes are known to be stable not only by experience, but also by laws of thermodynamics. However, the PDE we are

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investigating is not of standard linear type, but is extended by a nonlinear term $\kappa(T)T_x^2$, which might cause instability or lead to convergence issues. Therefore, a stability analysis is essential as by proving stability for the model (6) one can conclude that it can in fact be used to describe freezing processes with phase change.

The rest of the paper is organized as follows. Section 2 introduces the problem setting and the model, which has already been described in e.g Backi and Gravdahl (2013) and Backi et al. (2014b). In Section 3 the steady state solution to the model of Section 2 is presented. Section 4 provides the main stability results, whereas Section 5 shows a numerical example, which highlights the results in the previous section. Finally, Section 6 delivers some concluding remarks and comments on future work.

Notation: Let $L^2([0,1])$ denote the space of real-valued, square integrable functions f defined on $[0,1]$ with finite L^2-norm; $\|f\|^2 = \int_0^1 f(x)^2 dx < \infty$. The space $H^1([0,1])$ is the subspace of $L^2([0,1])$ consisting of functions g with finite H^1-norm; $\|g\|_{H^1} = \|g\|^2 + \|g_x\|^2 < \infty$. In this paper we deal with functions $w = w(t,x)$ of time t and space x (the spatial variable). To ease notation we frequently leave out the dependency on t and/or x, e.g., $\|w(t)\|$ is the L^2-norm of the function $x \mapsto w(t)(x) = w(t,x)$.

2. PROBLEM FORMULATION

The problem was originally presented in Backi and Gravdahl (2013) for an application that models and controls the freezing of fish in vertical plate freezers. The case considered there, as well as in Backi et al. (2014b) and Backi et al. (2014a) is also regarded in the present paper, where a parabolic PDE (diffusion equation) is formulated in the state variable T representing temperature, as follows:

$$\rho(T)\, c(T)\, T_t = [\lambda(T)\, T_x]_x \tag{1}$$

subject to Dirichlet boundary conditions

$$\begin{aligned} T(t,0) &= T_0 \\ T(t,L) &= T_L, \end{aligned} \tag{2}$$

where $\rho(T)$ describes the density, $c(T)$ denotes the specific heat capacity at constant pressure and $\lambda(T)$ indicates the thermal conductivity of the medium to be frozen. Note that $\rho(T) > 0$, $c(T) > 0$ and $\lambda(T) > 0$ can be regarded as thermodynamic alloys of substances like water, fat, etc. as described in Backi and Gravdahl (2012). The boundary conditions T_0 and T_L are given by the refrigerant temperatures at $x = 0$ and $x = L$, respectively. Furthermore be advised that the two subscripts $(\cdot)_t$ and $(\cdot)_x$ denote derivatives wrt time t and the spatial variable x, respectively.

We can rewrite (1) in the following form

$$\rho(T)\, c(T)\, T_t = \lambda_T(T)\, T_x^2 + \lambda(T)\, T_{xx}, \tag{3}$$

where we introduce two new parameters as

$$k(T) = \frac{\lambda(T)}{\rho(T)\, c(T)}, \tag{4}$$

$$\kappa(T) = \frac{\lambda_T(T)}{\rho(T)\, c(T)}. \tag{5}$$

This leads to a rewritten form of (3):

$$T_t = \kappa(T)\, T_x^2 + k(T)\, T_{xx}, \tag{6}$$

which is still subject to the boundary conditions in (2).

Figure 1 displays a qualitative sketch of parameter variations in $\lambda(T)$ and $c(T)$ over T. Note that the variation in $\rho(T)$ over T is of minor consequence and therefore assumed negligible, i.e. $\rho(T) = const$. The parameter functions were defined to approximate real parameter values sufficiently well. Furthermore, like mentioned in Section 1, they are adapted to model for the so-called *thermal arrest* caused by *latent heat of fusion*. This is achieved in particular by overestimating $c(T)$ in the region $I_{\Delta T}$ to slow down heat transfer and thus the desired behavior is obtained. We point out that the transitions from c_s to c_i and from c_i to c_l have not specifically been introduced in Backi et al. (2014a). In this paper, however, the transitions are considered to be functions of the shape $c(T) = \frac{p}{T+q}$ in small neighbourhoods outside $I_{\Delta T}$, where p and q are constants. In Backi et al. (2014b)

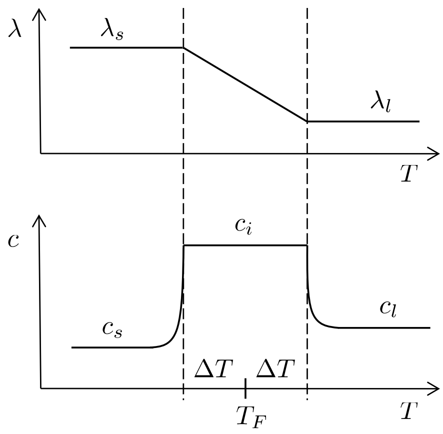

Fig. 1. Qualitative sketch of parameter variations in λ and c over T

the stability properties of (6) were shown to hold for equal boundary conditions $T_0 = T_L$ subject to some restrictions on the state-dependent parameter functions. Providing a generalization to the proof in Backi et al. (2014b), namely in the case when $T_0 \neq T_L$, motivates the subsequent investigation.

In the sequel, we shall refer to the following two well-known lemmas taken from Krstic and Smyshlyaev (2008):

Lemma 1. (Poincaré's Inequality) For any continuously differentiable function $\omega = \omega(z)$, the following inequalities hold:

$$\begin{aligned} \|\omega\|^2 &\le 2\omega(0)^2 + 4\|\omega_z\|^2, \\ \|\omega\|^2 &\le 2\omega(1)^2 + 4\|\omega_z\|^2. \end{aligned}$$

Lemma 2. (Agmon's Inequality) For any function $\omega = \omega(t,x)$ with $\omega(t) \in H^1([0,1])$, the following inequalities hold:

$$\begin{aligned} \max_{x\in[0,1]} |\omega(t,x)|^2 &\le \omega(0)^2 + 2\|\omega(t)\|\|\omega_x(t)\|, \\ \max_{x\in[0,1]} |\omega(t,x)|^2 &\le \omega(1)^2 + 2\|\omega(t)\|\|\omega_x(t)\|. \end{aligned}$$

3. STEADY-STATE SOLUTION

Before conducting the stability analysis of (6) we here derive an explicit formula for the steady-state solution to (6). This is done by setting $T_t = 0$, which results in

$$\kappa(T)\, T_x^2 + k(T)\, T_{xx} = 0. \tag{7}$$

For general $\kappa(T)$ and $k(T)$ the solution to the nonlinear ODE (7) can be found by evaluating the following expression:

$$C_1 x + C_2 = \int^{T(x)} \exp\Big(\int \frac{\kappa(z)}{k(z)} \, dz \Big) \, dz. \tag{8}$$

Now from physical considerations it follows that the steady state solution must be within the interval $[T_0, T_L]$ (or $[T_L, T_0]$ depending on which of T_0 and T_L is the smaller one) for all values of the spatial coordinate. Also, inspired by the qualitative sketch shown in Figure 1, we henceforth state that $\kappa(T) = 0$ outside $I_{\Delta T}$. This is valid as $\lambda(T)$ is assumed constant outside $I_{\Delta T}$. Based on these considerations and since we assume that the boundary conditions T_0 and T_L are strictly below $T_F - \Delta T$, we conclude that $\kappa(T)$ is zero along the steady state solution.

From (7) it then follows that the steady-state solution can be described as the solution of the ODE

$$k(T) T_{xx} = 0, \tag{9}$$

which, since $k(T) > 0$ for all T, has the solution

$$T(x) = C_1 x + C_2. \tag{10}$$

The coefficients C_1 and C_2 can be found by applying the boundary conditions (2) to (10), which leads to $C_2 = T_0$ and $C_1 = \frac{1}{L}(T_L - T_0)$.

Thus the steady-state solution has the form

$$T(x) = \frac{1}{L}(T_L - T_0)x + T_0, \tag{11}$$

which represents a straight line between the two boundary values T_0 and T_L.

4. STABILITY ANALYSIS

The PDE (6) is specific for the freezing application. As we intend to prove stability, however, we choose to take a more general view of the problem in this section and to indicate this, we change the state variable from T to u.

In general, the function u can be expressed as the sum of a transient part $w(t,x)$ and a stationary part $\bar{u}(x)$, i.e. $u(t,x) = w(t,x) + \bar{u}(x)$. Note that $\bar{u}(x)$ is a function of x due to the asymmetric boundary conditions (2). The spatial coordinate is normalized to belong to $[0,1]$ and the stationary part is required to be in the same form like (11), hence $\bar{u}(x) = Sx + R$, where S and R are constants. Thus we study the following equivalent form of (6):

$$\begin{aligned} w_t =& \frac{\kappa}{L^2}(w_x + S)^2 + \frac{k}{L^2} w_{xx} \\ =& \frac{\kappa}{L^2}(w_x^2 + S^2 + 2Sw_x) + \frac{k}{L^2} w_{xx} \end{aligned} \tag{12a}$$

with boundary conditions

$$w(t,0) = w(t,1) = 0. \tag{12b}$$

Remark 3. We must point out that our focus lies on continuously differentiable solutions with finite H^1-norm only. From a rigorous mathematical point of view the question of existence of such solutions is a crucial aspect, however, we will not approach that here (see Prüss et al. (2007) who provide a treatment of this for the (related) Stefan problem). Nevertheless, we point out that our application studies indicate that at least some solutions of this form exist.

With this remark we continue and state the following result, which is an extension to (Backi et al., 2014b, Lemma 3) for the case of asymmetric boundary conditions.

Lemma 4. Let w satisfy (12). Suppose that there exists constants $\beta > \alpha > 0$ such that $\alpha \le k \le \beta$. If

$$(\kappa + k_u)^2 < 2\left(\kappa k_u - k_{uu} k + k_u^2\right) \tag{13a}$$
$$k_{uu} k < k_u^2 + \kappa k_u \tag{13b}$$
$$w > 0 \; \forall u \in I_{\Delta u} \tag{13c}$$
$$\kappa k_u \ge 0 \tag{13d}$$
$$k_{uu} \le 0 \; \forall u \in I_{\Delta u} \tag{13e}$$
$$\kappa < 0 \; \forall u \in I_{\Delta u} \tag{13f}$$
$$\kappa \equiv 0 \; \forall u \notin I_{\Delta u} \tag{13g}$$

then the origin is globally asymptotically stable (wrt $\|\cdot\|$). In particular $\|w(t)\| \to 0$ as $t \to \infty$.

Proof. Define the Lyapunov candidate V by

$$V = \int_0^1 \frac{1}{k} w^2 \, dx \tag{14}$$

and note that

$$V \ge \frac{1}{\beta} \|w(t)\|^2 \tag{15}$$

since $k \le \beta$ by assumption.

Differentiating (14) with respect to time leads to

$$\begin{aligned} \dot{V} =& \int_0^1 \frac{2}{k} w w_t - \frac{k_u}{k^2} w^2 w_t \, dx \\ =& \frac{1}{L^2} \int_0^1 \Big(\frac{2\kappa}{k} w w_x^2 + \frac{2\kappa}{k} w S^2 + \frac{4\kappa}{k} w S w_x + 2 w w_{xx} \\ & - \frac{\kappa k_u}{k^2} w^2 w_x^2 - \frac{\kappa k_u}{k^2} w^2 S^2 - \frac{2\kappa k_u}{k^2} w^2 S w_x \\ & - \frac{k_u}{k} w^2 w_{xx} \Big) \, dx. \end{aligned} \tag{16}$$

Integrating the terms $w w_{xx}$ and $\frac{k_u}{k} w^2 w_{xx}$ by parts yields

$$\int_0^1 w w_{xx} \, dx = \Big[w w_x \Big]_0^1 - \int_0^1 w_x^2 \, dx \tag{17}$$

$$\begin{aligned} \int_0^1 \frac{k_u}{k} w^2 w_{xx} \, dx =& \Big[\frac{k_u}{k} w^2 w_x \Big]_0^1 - 2 \int_0^1 \frac{k_u}{k} w w_x^2 \, dx \\ & - \int_0^1 \frac{k_{uu} k - k_u^2}{k^2} w^2 w_x^2 \, dx \end{aligned} \tag{18}$$

with $\Big[w w_x \Big]_0^1 = 0$ and $\Big[\frac{k_u}{k} w^2 w_x \Big]_0^1 = 0$ due to (12b).

Then, after inserting (17) and (18) into (16) and collecting terms the following expression is obtained

$$\dot{V} = \frac{1}{L^2} \int_0^1 \frac{1}{k} \Big[A w_x^2 + B w_x + C \Big] \, dx, \tag{19}$$

where we have used the shorthand

$$
\begin{aligned}
A =& -w^2\frac{1}{k}(\kappa k_u - k_{uu}k + k_u^2) + w(2\kappa + 2k_u) - 2k && \text{(20a)}\\
B =& w^2\frac{1}{k}(-2\kappa k_u S) + w(4\kappa S) && \text{(20b)}\\
C =& w^2\frac{1}{k}(-\kappa k_u S^2) + w(2\kappa S^2). && \text{(20c)}
\end{aligned}
$$

Note that $A < 0$, since the coefficients

$$
\begin{aligned}
a =& -\frac{1}{k}(\kappa k_u - k_{uu}k + k_u^2) && \text{(21a)}\\
b =& 2\kappa + 2k_u && \text{(21b)}\\
c =& -2k && \text{(21c)}
\end{aligned}
$$

of the parabola $aw^2 + bw + c$ defined in (20a) fulfill $a < 0$ and $b^2 - 4ac < 0$, by assumptions (13d)-(13g).

The rest of the proof consists of the following two observations outside and inside $I_{\Delta T}$, respectively.

Firstly, by (13g) we infer that outside $I_{\Delta T}$ both $B = 0$ and $C = 0$. Hence, for this case, we can bound (19) using that $\frac{1}{\beta} \le \frac{1}{k}$ followed by applying Lemma 1 (*Poincaré's Inequality*), leading to

$$
\dot{V} \le \frac{K_1}{L^2\beta}\|w_x\|^2 \le \frac{K_1}{4L^2\beta}\|w\|^2 \tag{22}
$$

with $K_1 = \max(A) < 0$.

Secondly we note that the inequality $B^2 - 4AC < 0$ holds true inside $I_{\Delta T}$, as can be seen as follows. Using (20) the inequality $B^2 - 4AC < 0$ is equivalent to the following quartic inequality

$$
\begin{aligned}
&\frac{w^4}{k^4}(4\kappa k_u S^2(k_{uu}k - k_u^2)) + \frac{w^3}{k^3}(-8\kappa S^2(k_{uu}k - 2k_u^2))\\
&+ \frac{w^2}{k^2}(-24\kappa k_u S^2) + \frac{w}{k}(16\kappa S^2) < 0.
\end{aligned} \tag{23}
$$

Let

$$
\psi(w) = a_4w^4 + a_3w^3 + a_2w^2 + a_1w
$$

denote the 4th order polynomial defined by the left hand side of (23) and note that $a_i < 0$, $i = 1,2,3,4$ inside $I_{\Delta T}$ by the assumptions (13d)–(13f). Let $\phi(w)$ denote the 3rd order polynomial defined by $\psi(w) = w\phi(w)$. Then (23) holds inside $I_{\Delta T}$ iff $\phi(w) < 0$ for $w > 0$. Since both, $a_4 < 0$ and $a_1 < 0$ inside $I_{\Delta T}$, it is enough to show that the roots of $\phi(w)$ all are negative or complex, which follows from the fact that inside $I_{\Delta T}$ we have $a_4 < 0$, $\phi_w(0) = a_2 < 0$ and $\phi_{ww}(0) = a_3 < 0$. Hence as long as we are inside $I_{\Delta T}$ there exists a constant $K_2 < 0$ such that

$$
\dot{V} \le \frac{K_2}{L^2\beta}\|w_x\|^2 \le \frac{K_2}{4L^2\beta}\|w\|^2. \tag{24}
$$

By letting $\bar{K} = \max\{K_1, K_2\}$, both inequalities (22) and (24) now yield

$$
\dot{V} \le \frac{1}{4L^2\beta}\bar{K}\|w\|^2, \tag{25}
$$

which together with (14) and (Henry, 1981, Theorem 4.1.4) proves the lemma.

We now extend Lemma 4 to the H^1-case.

Lemma 5. Suppose that the assumptions of Lemma 4 hold true. If moreover

$$
\begin{aligned}
&\kappa_u \le 0 && \text{(26a)}\\
&w_x(t,1)w_{xx}(t,1) - w_x(t,0)w_{xx}(t,0) \le 0 && \text{(26b)}\\
&\kappa S\Big(w_x^2(t,1) - w_x^2(t,0)\Big) \le 0 && \text{(26c)}\\
&\kappa\Big(w_x^3(t,1) - w_x^3(t,0)\Big) \le 0 && \text{(26d)}
\end{aligned}
$$

then the origin is globally asymptotically stable (wrt $\|\cdot\|_{H^1}$). In particular $\|w(t)\|_{H^1} \to 0$ as $t \to \infty$.

Proof. Define the Lyapunov candidate Λ by

$$
\Lambda = V_1 + V = \frac{1}{2}\int_0^1 w_x^2\,dx + V \tag{27}
$$

where V denotes the Lyapunov function defined by (14).

The time derivative of V_1 is

$$
\dot{V}_1 = \int_0^1 w_x w_{tx}\,dx. \tag{28}
$$

To obtain an expression for w_{tx} in terms of spatial derivatives of w only, the derivative of (12) with respect to x is calculated and one obtains

$$
\begin{aligned}
w_{tx} = \frac{1}{L^2}\Big(&\kappa_u w_x^3 + 2\kappa_u S w_x^2 + \kappa_u S^2 w_x + 2\kappa w_x w_{xx}\\
&+ 2\kappa S w_{xx} + k_u w_x w_{xx} + k w_{xxx}\Big).
\end{aligned} \tag{29}
$$

Combining (29) and (28) and collecting terms gives

$$
\begin{aligned}
\dot{V}_1 = \frac{1}{L^2}\int_0^1 &\kappa_u w_x^4 + 2\kappa_u S w_x^3 + \kappa_u S^2 w_x^2 + 2\kappa w_x^2 w_{xx}\\
&+ 2\kappa S w_x w_{xx} + k_u w_x^2 w_{xx} + k w_x w_{xxx}\,dx.
\end{aligned} \tag{30}
$$

Integrating the terms $kw_x w_{xxx}$ and $\kappa w_x w_{xx}$ by parts yields

$$
\int_0^1 k w_x w_{xxx}\,dx = \Big[k w_x w_{xx}\Big]_0^1 - \int_0^1 k w_{xx}^2\,dx - \int_0^1 k_u w_x^2 w_{xx}\,dx \tag{31}
$$

$$
\begin{aligned}
\int_0^1 \kappa w_x w_{xx}\,dx &= \Big[\kappa w_x^2\Big]_0^1 - \int_0^1 \kappa w_x w_{xx}\,dx - \int_0^1 \kappa_u w_x^3\,dx\\
&= \Big[\frac{\kappa}{2}w_x^2\Big]_0^1 - \int_0^1 \frac{\kappa_u}{2}w_x^3\,dx.
\end{aligned} \tag{32}
$$

Putting (31) and (32) into (30) one obtains

$$
\begin{aligned}
\dot{V}_1 = \frac{1}{L^2}\int_0^1 &\kappa_u w_x^4 + \kappa_u S w_x^3 + \kappa_u S^2 w_x^2 + 2\kappa w_x^2 w_{xx} - k w_{xx}^2\,dx\\
&+ \frac{1}{L^2}\Big[k w_x w_{xx} + \kappa S w_x^2\Big]_0^1.
\end{aligned} \tag{33}
$$

Furthermore, integrating the expression $\kappa w_x^2 w_{xx}$ by parts gives

$$
\begin{aligned}
\int_0^1 \kappa w_x^2 w_{xx}\,dx &= \Big[\kappa w_x^3\Big]_0^1 - \int_0^1 2\kappa w_x^2 w_{xx}\,dx - \int_0^1 \kappa_u w_x^4\,dx\\
&= \Big[\frac{1}{3}\kappa w_x^3\Big]_0^1 - \int_0^1 \frac{1}{3}\kappa_u w_x^4\,dx.
\end{aligned} \tag{34}
$$

After substituting (34) into (33) we receive

$$
\begin{aligned}
\dot{V}_1 = \frac{1}{L^2}\int_0^1 &\frac{\kappa_u}{3}w_x^4 + \kappa_u S w_x^3 + \kappa_u S^2 w_x^2 - k w_{xx}^2\,dx\\
&+ \frac{1}{L^2}\Big[k w_x w_{xx} + \kappa S w_x^2 + \frac{2\kappa}{3}w_x^3\Big]_0^1.
\end{aligned} \tag{35}
$$

Now we need to have a closer look at the quartic equation $\theta(w_x) = dw_x^4 + ew_x^3 + fw_x^2$ with the shorthand

$$d = \frac{\kappa_u}{3}, \quad e = \kappa_u S, \quad f = \kappa_u S^2,$$

which can be rewritten as $w_x^2 \gamma(w_x)$. For the quadratic equation $\gamma(w_x)$ we must impose that it is less than zero for all w_x and thus it must hold that $d < 0$ and $e^2 - 4df \leq 0$. If now (26a) holds, we see that $\gamma(w_x) < 0$ for all w_x and therefore $\theta(w_x) < 0$.

If furthermore (26b)–(26d) hold, we can infer that

$$\dot{V}_1 \leq -\frac{1}{L^2}\int_0^1 k w_{xx}^2 \, dx \leq -\frac{\alpha}{L^2}\int_0^1 w_{xx}^2 \, dx. \tag{36}$$

Thus by putting (25) and (36) into the time-derivative of (27), using Lemma 1 (*Poincaré's Inequality*) and recalling that $\bar{K} < 0$, we receive

$$\begin{aligned}
\dot{\Lambda} &\leq -\frac{\alpha}{L^2}\int_0^1 w_{xx}^2 \, dx + \frac{\bar{K}}{L^2\beta}\int_0^1 w_x^2 \, dx \leq \frac{\bar{K}}{L^2\beta}\int_0^1 w_x^2 \, dx \\
&\leq \frac{\bar{K}}{2L^2\beta}\int_0^1 w_x^2 \, dx + \frac{\bar{K}}{2L^2\beta}\int_0^1 w_x^2 \, dx \\
&\leq \frac{\bar{K}}{8L^2\beta}\int_0^1 w^2 \, dx + \frac{\bar{K}}{2L^2\beta}\int_0^1 w_x^2 \, dx \\
&\leq \frac{\bar{K}}{8L^2\beta}\|w(t)\|_{H^1},
\end{aligned} \tag{37}$$

which, together with (27) and (Henry, 1981, Theorem 4.1.4) proves the lemma.

Together with Lemma 2 *(Agmon's Inequality)*, Lemma 5 now immediately implies the following main result of the paper.

Theorem 6. Let w satisfy (12). Suppose that the assumptions of Lemma 5 hold true. Then $w(t,x) \to 0$ as $t \to \infty$, hence $u(t,x) \to \bar{u}(x)$ as $t \to \infty$.

Remark 7. (Discussion about assumptions). The assumptions in Lemma 4 and Lemma 5 impose limitations on the parameter functions $k(u)$ and $\kappa(u)$ and their respective derivatives wrt u. In Lemma 4 assumptions (13d)–(13g) are satisfied due to the definition of the parameter functions in Figure 1. Assumption (13c) holds true due to the fact that the boundary conditions are chosen constant below the region $I_{\Delta u} = I_{\Delta T}$ and the definition of w in the beginning of this subsection. Assumptions (13a)–(13b) have to be imposed to the problem and are valid, also due to the definition of the parameter functions.
In Lemma 5 assumption (26a) is satisfied by the definition of the parameter functions, again see Figure 1. Assumptions (26b)–(26d) define conditions for the temperature change with respect to the spatial domain and its derivative, both evaluated at the respective boundaries.

5. SIMULATION EXAMPLES

For the simulations we now return to the original freezing application, where we have chosen asymmetric boundary conditions (BCs) and noisy initial conditions (ICs) in order to exemplify the theoretical results in Section 4. Simulation parameters can be found in Backi and Gravdahl (2013) and represent a physical freezing process. Second order central differences and first order forward and central differences have been used to discretize the PDE (6) in its spatial coordinates only. This resulted in a set of coupled ODEs representing a spatial resolution of approximately $1 \cdot 10^{-3}$ m ($N = 99$ discretization steps).

We present one case whose behavior has already been presented in Backi et al. (2014b). However, in Backi et al. (2014b) the convergence to the steady state value was not covered by the theorem due to its conservative formulation. Now we can state that the simulations presented in Backi et al. (2014b), as well as the ones illustrated below, are covered by Theorem 6.

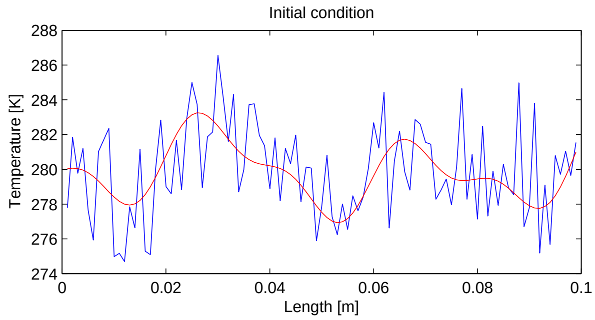

Fig. 2. Initial condition, red: sum of sinusoidals, blue: sinusoidals plus added white Gaussian noise

Figure 2 shows the noisy IC that was chosen for the simulations. It consists of a sum of sinusoidals of different frequencies around $T = 280$ K plus added white Gaussian noise. The red line illustrates the sinusoidals whereas the blue line the overall noisy IC. The damping character of the linear heat equation is well-known; By choosing the IC in this fashion we want to emphasize that the damping character still holds for (6), even in the presence of the term $\kappa(T)T_x^2$.

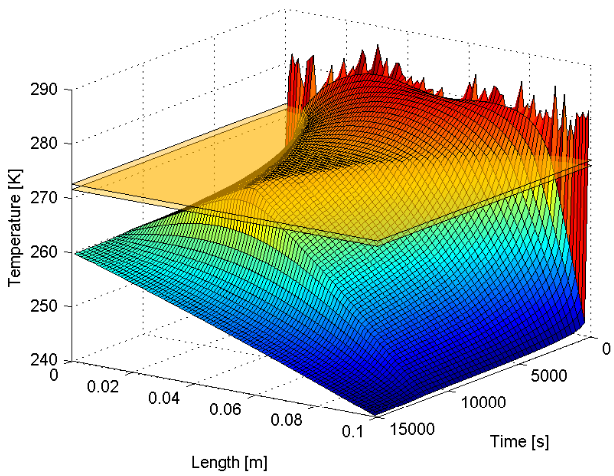

Fig. 3. Asymmetric BCs and noisy IC

In Figure 3 a simulation with BCs $T(t,0) = 260$ K and $T(t,0.1) = 240$ K is presented. We can see that the temperature distribution converges towards the expected steady state solution and is clearly stable in accordance with Theorem 6. Moreover, we can see the phenomenon of *thermal arrest*, which takes the form of a plateau of nearly constant temperatures inside $I_{\Delta T}$. The region $I_{\Delta T}$ is emphasized by the two orange rectangles. Furthermore, the *thermal arrest* is best visible in the very center of the spatial domain. The overall behavior corresponds with freezing curves obtained by measurements, as presented e.g. in Nicholson (1973).

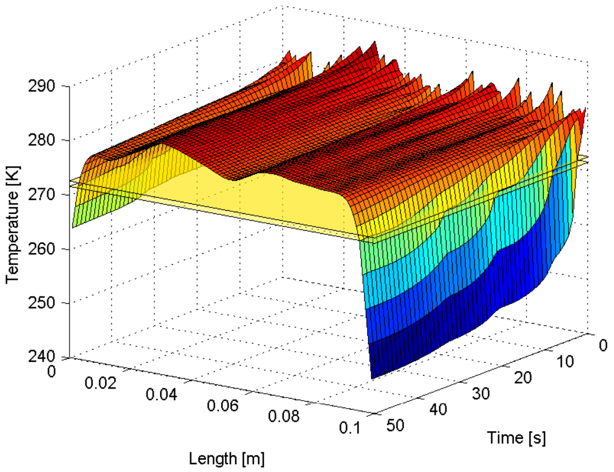

Fig. 4. Asymmetric BCs and noisy IC - 0 to 50 s

Figure 4 shows the behaviour in the first 50 s for the case shown in Figure 3. Like mentioned earlier, we do this in order to illustrate the converging and damping character even for noisy IC. As can be seen, the low-frequency parts of the sinusoidals are still present at the end of the simulation time, whereas the high-frequency peaks caused by white Gaussian noise get levelled out fairly fast.

6. CONCLUSIONS AND FUTURE WORK

In this paper we presented a stability investigation for a partial differential equation with state-dependent parameter functions and asymmetric boundary conditions. The PDE is a heat equation derived from the diffusion equation. The work is specific for a freezing application and presents a generalization to already established stability results for the same heat equation with symmetric boundary conditions.

Numerical simulations indicate that the theoretical results in Section 4 in fact hold for the freezing application. As pointed out in Backi et al. (2014b) we firstly proved stability in the sense of convergence in both L^2- and H^1-norms, and secondly in terms of absolute value of the solution's transient part. We were forced to impose restrictions on derivatives and signs of the coefficient functions.

Regarding future work, it was brought to our attention by comments of well-regarded colleagues that, by using *Friedrichs' Inequality* instead of *Poincaré's Inequality*, one can potentially find tighter bounds for the respective time derivatives of the Lyapunov candidates V and Λ.

Furthermore, one could investigate stability for different types of boundary conditions. Particularly, replacing the Dirichlet boundary conditions by those of Neumann type, representing heat flux at the respective boundaries. This is interesting from a mathematical point of view, however, it does not seem practical for the specific freezing case in vertical plate freezers. This is due to the fact that the heat flux is proportional to the temperature difference at the boundaries. A correct estimation of this proportionality factor, however, is hard to obtain, as it is mainly dictated by the flow regime and the spatially-dependent vapor quality (i.e. the mass fraction of the vapor phase) of the refrigerant.

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